3766. [2012 : 285, 287] *Proposed by M. A. Alekseyev.*

Let $x_1 < x_2 < \cdots < x_n$ be positive integers such that

$$\left(\sum_{k=1}^{n} x_k\right)^2 = \sum_{k=1}^{n} x_k^3 .$$

Prove that $x_k = k$ for each $k = 1, 2, \ldots, n$.

Solved by A. Alt; AN-anduud Problem Solving Group; M. Bataille; S. Malikić; D. Smith; E. Swylan; D. Văcaru; and the proposer. We present the solution by Arkady Alt.

We establish the stronger result that, if $\{x_k : k \geq 1\}$ is a strictly increasing sequence of positive integers, then for each positive integer n,

$$x_1^3 + x_2^3 + \cdots + x_n^3 \geq (x_1 + x_2 + \cdots + x_n)^2$$

with equality if and only if $x_k = k$ for $1 \leq k \leq n$.

Observe that, if $x_0 = 0$ and $i \geq 2$, then $x_i \geq x_{i-1} + 1$ and $x_{i-2} \leq x_{i-1} - 1$, so that

$$x_i x_{i-1} - x_{i-1} x_{i-2} \geq (x_{i-1}^2 + x_{i-1}) - (x_{i-1}^2 - x_{i-1}) = 2x_{i-1}.$$

This implies that, for $k \geq 2$,

$$x_k x_{k-1} = \sum_{i=2}^{k} (x_i x_{i-1} - x_{i-1} x_{i-2}) \geq 2 \sum_{i=2}^{k} x_{i-1}.$$

Equality occurs if and only if $x_i = x_{i-1} + 1$, *i.e.* $x_i = i$ for $1 \leq i \leq k$.

For each positive integer n,

$$\begin{aligned}\sum_{k=1}^{n} x_k^3 &\geq \sum_{k=1}^{n} x_k^2 (1 + x_{k-1}) = \sum_{k=1}^{n} x_k^2 + \sum_{k=1}^{n} x_k (x_k x_{k-1}) \\ &= \sum_{k=1}^{n} x_k^2 + 2 \sum_{k=2}^{n} x_k (x_1 + x_2 + \cdots + x_{k-1}) = \left(\sum_{k=1}^{n} x_k\right)^2 ,\end{aligned}$$

with equality if and only if $x_k = k$ for $1 \leq k \leq n$.

Editor's note. There were three flawed solutions that sought to prove the equality by induction. They failed to consider the possibility that the truth of the equation in the problem for $n = m + 1$ need not entail that it holds for any m of the integers involved.

Bataille built his solution on the identity

$$\sum_{k=1}^{n} x_k^3 - \left(\sum_{k=1}^{n} x_k\right)^2 = \sum_{k=1}^{n} x_k \sum_{j=1}^{k} (x_j + x_{j-1})(x_j - x_{j-1} - 1).$$

It is likely that this problem goes back a long way. Malikić pointed out that a version of it appeared on page 135 of Volume 38:4 (April, 2012) of this journal as problem OC16, with a solution due to Titu Zvonaru of Comanesti, Romania. Mihaly Bencze noted that he published a solution to it in an article appearing in Octagon Mathematical Magazine 6:2 (October, 1998), 110-115. There are vastly many sets of integers for which the sum of the cubes is equal to the square of the sum, even for as few as three elements, when we allow repetitions and negative integers. The 2013 paper Sum of cubes is square of sum by Samer Seraj and Edward Barbeau (`arxiv.org/pdf/1306.5757v1.pdf`) explores this fecund area and includes Seraj's proof of the inequality of our solution. Earlier references to the problem are welcome.

3767. [2012 : 285, 287] *Proposed by D. Milošević.*

Let R, r be the circumradius and inradius of a right-angled triangle. Prove that

$$\frac{R}{r} + \frac{r}{R} \geq 2\sqrt{2}\,.$$

Solved by A. Alt; AN-anduud Prolem Solving Group; G. Apostolopoulos; Š. Arslanagić; D. Bailey, E. Campbell and C. R. Diminnie; M. Bataille; B. D. Beasley; C. Curtis; M. Dincă; J. Hawkins and D. Stone; R. Hess; V. Konečný; D. Koukakis; K. Lau; S. Malikić; C. M. Quang; C. Sánchez-Rubio; E. Suppa; E. Swylan; I. Uchiha; D. Văcaru; H. Wang and J. Wojdylo; P. Y. Woo; T. Zvonaru; and the proposer. We present 2 solutions.

Solution 1 by Brian D. Beasley.

Since the function $f(x) = x + \dfrac{1}{x}$ is increasing on $[1, \infty)$, it suffices to show that

$$\frac{r}{R} \leq \sqrt{2} - 1,$$

as then

$$\frac{R}{r} \geq \sqrt{2} + 1$$

and hence

$$\frac{R}{r} + \frac{r}{R} = f\left(\frac{R}{r}\right) \geq f(\sqrt{2}+1) = 2\sqrt{2}.$$

For a right triangle, we have $r = ab/(a+b+c)$ and $R = c/2$. Then

$$\frac{r}{R} = \frac{2ab}{c(a+b+c)} = \frac{a+b-c}{c},$$

since $c(a+b+c)(a+b-c) = c(a^2+2ab+b^2-c^2) = 2abc$. Thus to establish that $r/R \leq \sqrt{2}-1$, we must show that $(a+b)/c \leq \sqrt{2}$, or equivalently that $2ab \leq c^2$.